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Fermi Substructure of Space-Time

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If space-time possesses a Fermi substructure, then the canonical quantization of the space-time and the Fermi coordinates of a relativistic point particle must be mutually consistent. We show that the Fermi substructure $x^{\mu} = \frac{1}{4}\sigma^{\mu}_{A\dot{B}}\{c^{A}, c^{*\dot{B}}\}$ meets this requirement. We express the generators of the Lorentz group in terms of the Fermi coordinates and momenta and consider their coordinate representation.

1. CANONICAL EQUATIONS

Penrose (1967) pointed out that there is evidence of a direct connection between quantum mechanics and the structure of space-time, for example, in the elementary fact that different spatial directions of the spin of a spin-one-half particle correspond to taking different complex linear combinations of the two quantum states.

Considerations like these naturally raise the question as to whether space-time possesses a complex substructure which reflects the complex properties of both the Lorentz group and quantum mechanics. We would expect such a complex substructure to be associated with a Fermi system. Fermi substructures of space-time have been discussed in Schwarz and Van Nieuwenhuizen (1982) and Borchsenius (1987, 1989). In the following we suggest that the canonical quantization of a relativistic point particle can be used as a model to determine a Fermi substructure of space-time.

Following Dirac (1950, 1958), we take the space-time coordinates of the particle to be functions of a parameter, time, and quantize them according to

$$x^{\mu} \to X^{\mu}, \qquad X^{\mu\dagger} = X^{\mu} \tag{1}$$

The canonical momenta P_{μ} are defined through the Lagrangian

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$$\delta L = P_\mu \, \delta \dot{X}^\mu \tag{2}$$

and X and P satisfy the commutation relations

$$[X^\mu, P_\nu] = i\hbar \delta^\mu_\nu \tag{3a}$$

$$[X^\mu, X^\nu] = 0, \qquad [P_\mu, P_\nu] = 0 \tag{3b}$$

If the space-time coordinates arise from underlying Fermi coordinates, then the quantization (1) must result from a corresponding quantization of these. The simplest representation of the canonical Fermi coordinates, and the one which we shall consider here, has components C^A_a which transform like a right-handed two-component spinor in the index A and like a quantum ket vector in the index a. To distinguish C^A_a from a quantum operator, we shall write it in abstract form as $\mathbf{C}^A$, and its conjugate bra vector as $\mathbf{C}^{\dagger \dot{A}}$. Since the quantum vectors which we consider have noncommutative components, we shall need to form commutators between them. The anticommutators between a ket vector $\boldsymbol{\chi}$ and a bra vector $\boldsymbol{\psi}$ will be defined as

$$\{\boldsymbol{\chi}, \boldsymbol{\psi}^\dagger\}_{ab} \overset{\text{def}}{=} \chi_a \psi^*_b + \psi^*_b \chi_a \tag{4a}$$

$$\{\boldsymbol{\Psi}^\dagger, \boldsymbol{\chi}\} \overset{\text{def}}{=} \psi^*_a \chi_a + \chi_a \psi^*_a \tag{4b}$$

that is, we adopt the convention that the order in which the bra and ket vectors are written in the commutator determines whether *both* terms in the commutator are direct products or contractions.

$\mathbf{C}^A$ is accompanied by conjugate momenta $\mathbf{D}^\dagger_A$ defined through the Lagrangian:

$$\delta \operatorname{Tr} L = \{\mathbf{D}^\dagger_A, \delta \dot{\mathbf{C}}^A\} + \text{c.c.} \tag{5}$$

Corresponding to the space-time canonical system (3) we shall assume that $\mathbf{C}$ and $\mathbf{D}^\dagger$ satisfy the canonical anticommutation relations

$$\{\mathbf{C}^A, \mathbf{D}^\dagger_B\}\delta^{\dot{E}}_{\dot{F}} - \{\mathbf{D}_{\dot{F}}, \mathbf{C}^{\dagger \dot{E}}\}\delta^A_B = 2i\hbar \delta^A_B \delta^{\dot{E}}_{\dot{F}} \tag{6a}$$

$$\{\mathbf{C}^A, \mathbf{C}^B\} = 0 \tag{6b}$$

$$\{\mathbf{D}^\dagger_A, \mathbf{D}^\dagger_B\} = 0 \tag{6c}$$

We seek a relationship between the X and the $\mathbf{C}$ which will make the Fermi system (6) consistent with the space-time system (3). We shall show that if the space-time coordinates are determined by the Fermi coordinates through the relation

$$X^{\mu} = \frac{1}{4}\sigma^{\mu}_{A\dot{B}}\{\mathbf{C}^A, \mathbf{C}^{\dagger\dot{B}}\}, \qquad \{\mathbf{C}^A, \mathbf{C}^B\} = 0 \tag{7}$$

then the Fermi system (6) is a consequence of the space-time system (3). By differentiating (7) we obtain for the variation of $\dot{X}$ with respect to $\dot{\mathbf{C}}$

$$\delta\dot{X}^{\mu} = \frac{1}{4}\sigma^{\mu}_{B\dot{D}}(\{\delta\dot{\mathbf{C}}^B, \mathbf{C}^{\dagger\dot{D}}\} + \{\mathbf{C}^B, \delta\dot{\mathbf{C}}^{\dagger\dot{D}}\}) \tag{8}$$

When applied to the trace of (2) this gives

$$\delta \operatorname{Tr} L = \left\{\frac{1}{4}\sigma^{\mu}_{B\dot{D}}\mathbf{C}^{\dagger\dot{D}}P_{\mu}, \delta\dot{\mathbf{C}}^B\right\} + \text{c.c.} \tag{9}$$

When compared to (5) this yields the expression

$$\mathbf{D}^{\dagger}_{B} = \frac{1}{4}\sigma^{\mu}_{B\dot{D}}\mathbf{C}^{\dagger\dot{D}}P_{\mu} \tag{10}$$

for the Fermi momenta. To show that (6) follows from (3), we first observe that the anticommutativity (6b) of the Fermi coordinates is contained in the defining relation (7). The anticommutativity (6c) of the Fermi momenta follows directly from the anticommutativity of the $\mathbf{C}$ because in the expression (10) for $\mathbf{D}^{\dagger}$ the matrix elements of P_{μ} are commutative complex numbers. To show that (6a) is a consequence of (3a), we shall express the l.h.s. of (6a) in terms of X and P. To do this, we first rewrite (6a) in the more convenient form

$$\{\mathbf{C}^A, \mathbf{D}^{\dagger}_{B}\}\sigma^{B\dot{E}}_{\alpha}\sigma^{\beta}_{A\dot{E}} - \{\mathbf{D}_{\dot{B}}, \mathbf{C}^{\dagger\dot{A}}\}\sigma^{\dot{B}E}_{\alpha}\sigma^{\beta}_{\dot{A}E} = 4i\hbar\delta^{\beta}_{\alpha} \tag{11}$$

The anticommutator $\{\mathbf{C}, \mathbf{D}^{\dagger}\}$ can be expressed in terms of X and P by inserting the expression (10) for $\mathbf{D}^{\dagger}$ and applying the defining relation (7)

$$\begin{aligned}\{\mathbf{C}^A, \mathbf{D}^{\dagger}_{B}\} &= \{\mathbf{C}^A, \tfrac{1}{4}\sigma^{\mu}_{B\dot{D}}\mathbf{C}^{\dagger\dot{D}}P_{\mu}\} \\ &= \tfrac{1}{2}\sigma^{\mu}_{B\dot{D}}\sigma^{A\dot{D}}_{\nu}X^{\nu}P_{\mu}\end{aligned} \tag{12}$$

Using the property of the Pauli matrices

$$\sigma^{A\dot{B}}_{\nu}\sigma^{\mu}_{C\dot{B}}\sigma^{C\dot{D}}_{\alpha}\sigma^{\beta}_{A\dot{D}} = 2(\delta^{\mu}_{\nu}\delta^{\beta}_{\alpha} + \delta^{\mu}_{\alpha}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\eta_{\alpha\nu}) \tag{13}$$

(12) yields the remarkably simple expression for the l.h.s. of (11)

$$\begin{aligned}&\{\mathbf{C}^A, \mathbf{D}^{\dagger}_{B}\}\sigma^{B\dot{E}}_{\alpha}\sigma^{\beta}_{A\dot{E}} - \{\mathbf{D}_{\dot{B}}, \mathbf{C}^{\dagger\dot{A}}\}\sigma^{B\dot{E}}_{\alpha}\sigma^{\beta}_{A\dot{E}} \\ &= (\delta^{\mu}_{\nu}\delta^{\beta}_{\alpha} + \delta^{\mu}_{\alpha}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\eta_{\alpha\nu})[X^{\nu}, P_{\mu}]\end{aligned} \tag{14}$$

Inserting the expression for $[X, P]$ in (3a) into the r.h.s. of (14), we obtain (11) and thereby (6a). ■

A Hamiltonian H can be defined from the constraint which is associated with the free choice of parameter, time (Dirac, 1950, 1958). X^μ and P_μ then satisfy the equations of motion

$$\dot{X}^\mu = \frac{1}{i\hbar}[X^\mu, H], \qquad \dot{P}_\mu = \frac{1}{i\hbar}[P_\mu, H] \tag{15}$$

Applying these equations to (7) and (10), we find the equations of motion for the Fermi canonical variables

$$\dot{\mathbf{C}}^A = -\frac{1}{i\hbar} H\mathbf{C}^A, \qquad \dot{\mathbf{D}}_{\dot{A}} = -\frac{1}{i\hbar} H\mathbf{D}_{\dot{A}} \tag{16}$$

(6) and (16) together constitute the Fermi canonical system.

If the Fermi substructure is to reproduce all of space-time, (7) must not impose any constraint on X^μ apart of course from Hermicity. To prove that any Hermitian X^μ can be expressed in the form (7), we consider an arbitrary Hermitian matrix with the components M_{nm}. Such a matrix can be expressed in the form

$$M_{rs} = L_{rt}\lambda_t L^*_{st} \tag{17}$$

where the eigenvalues λ can be normalized to any of the values -1, 0, 1 by a rescaling of L. Consider the complex Clifford algebra

$$\{e_r, e^*_s\} = \delta_{r(s)}\lambda_{(s)}, \qquad \{e_r, e_s\} = 0 \tag{18}$$

where the λ's are the same as in (17). When the λ's contain all three values -1, 0, 1, the algebra is called indefinite and degenerate. Define the elements

$$a_r = L_{rt}e_t \tag{19}$$

where L is the same as in (17). Then it follows that M can be written as

$$M_{rs} = \{a_r, a^*_s\}, \qquad \{a_r, a_s\} = 0 \tag{20}$$

To apply this general result to X^μ, we observe that the spinorial components of X^μ satisfy

$$(X^{A\dot{B}}_{ab})^* = X^{B\dot{A}}_{ba}, \qquad X^{A\dot{B}}_{ab} \overset{\text{def}}{=} \sigma^{A\dot{B}}_\mu X^\mu_{ab} \tag{21}$$

and therefore can be considered as the components of a Hermitian matrix in the combined indices (A, a) and (B, b). Hence, according to the general result (20), $X^{A\dot{B}}_{ab}$ can be expressed in the form

$$X^{A\dot{B}}_{ab} = \tfrac{1}{2}\{C^A_a, C^{*\dot{B}}_b\}, \qquad \{C^A_a, C^B_b\} = 0 \tag{22}$$

These equations are equivalent to (7), provided the quantum conjugation †

performs the complex involution * of the Clifford algebra to which the C_a^A belong.

In the classical limit, (7) reduce to

$$x^\mu = \frac{1}{4}\,\sigma^\mu_{A\dot{B}}\{c^A, c^{*\dot{B}}\}, \qquad \{c^A, c^B\} = 0 \tag{23}$$

which defines the Fermi substructure of space-time itself. Since the commutation properties of the c's arise from quantum mechanics, we can say that, given the model discussed here, space-time is not merely a background space for quantum mechanics, but is itself a by-product of an underlying quantum structure.

2. INTERPRETATION OF THE QUANTUM AMPLITUDES

To understand the significance of the Fermi coordinates for the relationship between the complex properties of the Lorentz group and quantum mechanics, we shall consider the expectation values of X and $\mathbf{C}$.

First we express X and $\mathbf{C}$ in terms of the eigenstates of X:

$$X^\mu = |x_r^{(\mu)}\rangle x_r^{(\mu)}\langle x_r^{(\mu)}| \tag{24a}$$

$$\mathbf{C}^A = |x_r^\mu\rangle c_r^A, \qquad c_r^A \overset{\text{def}}{=} \langle x_r^\mu|\mathbf{C}^A \tag{24b}$$

where x_r^μ are the eigenvalues of X^μ. By inserting (24) into (7) we find that c_r^A satisfy

$$x_r^\mu = \frac{1}{4}\,\sigma^\mu_{A\dot{B}}\{c^A_{(r)}, c^{*\dot{B}}_{(r)}\} \tag{25}$$

A comparison of (25) with (23) shows that c_r^A are the Fermi coordinates corresponding to x_r^μ.

From (7) we obtain the expression for the expectation values $\bar{x}^\mu$ of X^μ

$$\bar{x}^\mu = \langle s|X^\mu|s\rangle = \frac{1}{4}\,\sigma^\mu_{A\dot{B}}\{\langle s|\mathbf{C}^A, \mathbf{C}^{\dagger\dot{B}}|s\rangle\} \tag{26}$$

Defining

$$\bar{c}^A \overset{\text{def}}{=} \langle s|\mathbf{C}^A \tag{27}$$

we can write (26) as

$$\bar{x}^\mu = \frac{1}{4}\,\sigma^\mu_{A\dot{B}}\{\bar{c}^A, \bar{c}^{*\dot{B}}\} \tag{28}$$

Hence $\bar{c}^A$ are the Fermi coordinates corresponding to $\bar{x}^\mu$. We shall call them the expectation values of $\mathbf{C}^A$. By inserting (24) into the r.h.s. of (26) and (27) we obtain the expressions

$$\bar{x}^\mu = |\langle s|x_r^{(\mu)}\rangle|^2 x_r^{(\mu)} \tag{29a}$$

$$\bar{c}^A = \langle s|x_r^\mu\rangle c_r^A \tag{29b}$$

for the expectation values of X and $\mathbf{C}$ in terms of the coordinate values which can result from a measurement.

Expression (29) shows that there is a parallel between the Fermi coordinates as a complex substructure of the space-time coordinates and the amplitudes as a complex "substructure" of the probabilities.

Expression (29b) leads to an interpretation of the quantum amplitudes as the complex weights with which the individual Fermi coordinates contribute to the expectation values $\bar{c}^A$. Since the Fermi coordinates owe their complexity to $SL(2.C)$, the complex property of the amplitudes is hereby being related to the structure of space-time.

3. FERMI FORM OF THE LORENTZ GENERATORS

The generators

$$L_{\mu\nu} = X_\nu P_\mu - X_\mu P_\nu \tag{30}$$

of the Lorentz group form a skew-symmetric tensor and are therefore equivalent to a symmetric second-rank spinor λ_{AB},

$$L_{A\dot{B}C\dot{D}} = \epsilon_{AC}\lambda^\dagger_{\dot{B}\dot{D}} + \epsilon_{\dot{B}\dot{D}}\lambda_{AC}, \qquad L_{A\dot{B}C\dot{D}} \stackrel{\text{def}}{=} \sigma^\mu_{A\dot{B}}\sigma^\nu_{C\dot{D}}L_{\mu\nu} \tag{31}$$

where

$$\lambda_{AC} = \tfrac{1}{2}\epsilon^{\dot{B}\dot{D}}L_{A\dot{B}C\dot{D}} \tag{32}$$

λ_{AC} satisfies an $SU(2)$ algebra and is the spinor form of the well-known non-Hermitian combination of the rotation and boost generators which is used to label the representations of the Lorentz group.

From (32) and (12) we obtain the expression

$$\lambda_{AB} = \{\mathbf{C}_A, \mathbf{D}^\dagger_B\} + \{\mathbf{C}_B, \mathbf{D}^\dagger_A\} \tag{33}$$

for λ_{AB} in terms of the Fermi coordinates and momenta. When the representation

$$X^\mu \to x^\mu, \qquad P_\mu \to -i\hbar\partial_\mu \tag{34}$$

is inserted into the r.h.s. of (32) we obtain, after a rearrangement which makes use of (23),

$$\lambda_{AB} \to \{c_A, -\tfrac{1}{4}\, i\hbar c^{*\dot{E}}\sigma^{\mu}_{\dot{B}\dot{E}}\partial_\mu\} + \{c_B, -\tfrac{1}{4} i\hbar c^{*\dot{E}}\sigma^{\mu}_{\dot{A}\dot{E}}\partial_\mu\} \tag{35}$$

A comparison of (35) with (33) shows that this representation can be obtained through the substitution

$$\mathbf{C}^A \to c^A, \qquad \mathbf{D}^{\dagger}_A \to -\tfrac{1}{4} i\hbar c^{*\dot{E}}\sigma^{\mu}_{\dot{A}\dot{E}}\partial_\mu \tag{36}$$

The Fermi expression (33) for the Lorentz generators invites considering the existence of coordinate representations of half-integer spin states. To obtain such representations, the expression (36) for $\mathbf{D}^{\dagger}_A$ must be generalized to act on functions of the Fermi coordinates which are not single-valued functions of the space-time coordinates. To this end we note that, according to (36), $\mathbf{D}^{\dagger}_A$ acts as a generator of displacements of the Fermi coordinates

$$\begin{aligned} \delta^{\psi}(x) &= \delta x^{\mu}\partial_{\mu}\psi \\ &= -\frac{1}{i\hbar}\left(\{\delta c^A, \mathbf{D}^{\dagger}_A(\psi(x))\} - \{\delta c^{*\dot{B}}, \mathbf{D}_{\dot{B}}(\psi(x))\}\right) \end{aligned} \tag{37}$$

The action of $\mathbf{D}^{\dagger}_A$ on a general function of the Fermi coordinates should therefore be defined correspondingly through

$$\delta\psi(c, c^*) = -\frac{1}{i\hbar}\left(\{\delta c^A, \mathbf{D}^{\dagger}_A(\psi(c, c^*))\} - \{\delta c^{*\dot{B}}, \mathbf{D}_{\dot{B}}(\psi(c, c^*))\}\right) \tag{38}$$

With this generalized definition of $\mathbf{D}^{\dagger}_A(\psi)$, we find by use of (31) and (33) that the azimuthal rotations of the Fermi coordinates

$$c \to \mathbf{e}^{-i\varphi/2}\sigma^3 c \tag{39}$$

are generated by the z-component of the angular momentum operator

$$\delta\psi(c, c^*) = -\frac{1}{i\hbar}\,\delta\varphi\, J_z(\psi(c, c^*)) \tag{40}$$

Accordingly, J_z has the representation

$$J_z = -i\hbar\,\frac{\partial}{\partial\varphi} \tag{41}$$

The condition for the eigenfunctions $e^{im\varphi}$ of J_z to be single-valued functions of the Fermi coordinates is that a rotation through an angle of 4π, which restores the value of c, must be a multiple of the period of $e^{im\varphi}$. This gives

$$m = 0, \pm\tfrac{1}{2}, \pm 1, \pm 3/2, \ldots \tag{42}$$

The eigenfunctions corresponding to half-integer spin are seen to be double-valued functions of the space-time coordinates. As expected, these states therefore have a coordinate representation only in terms of the Fermi coordinates.

The double homomorphism between the groups $SL(2.C)$ and $SO(1.3)$ associated with the Fermi and space-time coordinates, respectively, implies that there would appear to be a twofold degeneracy in space-time when viewed from Fermi space. It is interesting to speculate as to whether this feature has any significance for the interpretation of quantum mechanics, in particular whether it can provide a basis for the so-called "double space-time" interpretation of quantum mechanics (Bialynicki-Birula, 1986).

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